

SOME STUDIES ON LOEWY LENGTHS OF CENTERS OF p -BLOCKS

YOSHIHIRO OTOKITA

ABSTRACT. In this paper, we investigate some relations between the Loewy lengths of the centers of blocks of group algebras and its defect groups. In particular, we give a new upper bound of the Loewy length and determine the structure of blocks with large Loewy length.

1. INTRODUCTION

Let p be a prime, G a finite group and (K, \mathcal{O}, F) a splitting p -modular system for G where \mathcal{O} is a complete discrete valuation ring with quotient field K of characteristic 0 and residue field F of characteristic p . Passman has shown an upper bound for the Loewy length $LL(\mathbf{Z}FG)$ of the center of group algebra FG .

Theorem 1.1 (Passman [10]).

$$LL(\mathbf{Z}FG) \leq (p^{a+1} - 1)/(p - 1)$$

where $|G| = p^a m, p \nmid m$.

In the following, let B be a block of FG with a defect group D . Okuyama has generalized Theorem 1.1 for the center $\mathbf{Z}B$ of B .

Theorem 1.2 (Okuyama [8]).

$$LL(\mathbf{Z}B) \leq |D|$$

with equality if and only if D is cyclic and B is nilpotent.

In this paper, we investigate some relations between B, D and $LL(\mathbf{Z}B)$. In the next section, we show some fundamental properties and some examples of them. The third section deals with a new upper bound below in terms of the order and exponent of D .

Theorem 1.3. *Let p^d and p^ε be the order and exponent of D , respectively. Then we have*

$$LL(\mathbf{Z}B) \leq p^d - p^{d-\varepsilon} + 1.$$

In the last section, we determine the structure of blocks with large Loewy length as follows (In the following, we denote by C_m a cyclic group of order m and by $C_m \times C_n$ a direct product of two cyclic groups).

Theorem 1.4. *If $LL(\mathbf{Z}B) = |D| - 1$, then one of the following holds:*

- (1) $D \simeq C_3$.
- (2) $D \simeq C_2 \times C_2$ and B is Morita equivalent to FD .

Theorem 1.5. *If $LL(\mathbf{Z}B) = |D| - 2$, then one of the following holds:*

- (1) $D \simeq C_5$.
- (2) $D \simeq C_2 \times C_2$ and B is Morita equivalent to FA_4 .
- (3) $D \simeq C_2 \times C_2$ and B is Morita equivalent to the principal block of FA_5 .

where A_4 and A_5 are four and five degree alternating groups, respectively.

Theorem 1.6. *If $LL(\mathbf{Z}B) = |D| - 3$, then one of the following holds:*

- (1) $D \simeq C_5$.
- (2) $D \simeq C_7$.
- (3) $D \simeq C_4 \times C_2$ and B is Morita equivalent to FD .

By Theorem 1.2 and the results of Broué-Puig [1] and Puig [11], if $LL(\mathbf{Z}B) = |D|$, then B is Morita equivalent to FD where D is cyclic. So we have determined the structure of blocks with $|D| - 3 \leq LL(\mathbf{Z}B) \leq |D|$.

2. SOME FUNDAMENTAL RESULTS

In this paper, we denote by $k(B)$ and $l(B)$ the numbers of irreducible ordinary and Brauer characters associated to B , respectively. The next proposition is clear by the fact that $Z(B)$ is local.

Proposition 2.1. *The following are equivalent:*

- (1) D is trivial.
- (2) $LL(\mathbf{Z}B) = 1$.

Moreover, we give an upper bound of $LL(\mathbf{Z}B)$ in terms of $k(B)$ and $l(B)$.

Proposition 2.2.

$$LL(\mathbf{Z}B) \leq k(B) - l(B) + 1.$$

Proof. Let denote by $\mathbf{S}B$ and $\mathbf{S}\mathbf{Z}B$ the socle of B and $\mathbf{Z}B$, respectively. Then $k(B) = \dim Z(B)$, $l(B) = \dim \mathbf{S}B \cap \mathbf{Z}B$ and $\mathbf{S}B \cap \mathbf{Z}B \subseteq \mathbf{S}\mathbf{Z}B$ in general. Thus we have

$$\begin{aligned} LL(\mathbf{Z}B) - 1 &\leq \dim \mathbf{Z}B - \dim \mathbf{S}\mathbf{Z}B \\ &\leq \dim \mathbf{Z}B - \dim \mathbf{S}B \cap \mathbf{Z}B \\ &= k(B) - l(B), \end{aligned}$$

as required. \square

In the following, let β be a root of B , that is, a block of $F[DC_G(D)]$ such that $\beta^G = B$. We denote by $N_G(D, \beta)$ the inertial group of β in $N_G(D)$, by $I(B)$ the inertial quotient $N_G(D, \beta)/DC_G(D)$ and by $e(B) = |I(B)|$ the inertial index of B . In case D is cyclic, the Loewy length is given in [5].

Proposition 2.3 ([5, Corollary 2.8]). *If D is cyclic, then*

$$LL(\mathbf{Z}B) = \frac{|D| - 1}{e(B)} + 1.$$

In the following, $\mathbf{J}A$ denotes the Jacobson radical of an algebra A over F . The remainder of this section is devoted to some blocks with abelian defect group $D \simeq C_{p^m} \times C_{p^n}$ for some $m, n \geq 1$. These results are applied to the proof of our main theorems in the last section. First of all, we show the next lemma.

Lemma 2.4. *If D is normal in G , then $LL(\mathbf{Z}B) \leq p^m + p^n - 1$. In particular, if B is perfect isometric to its Brauer correspondent in $N_G(D)$, then $LL(\mathbf{Z}B) \leq p^m + p^n - 1$.*

Proof. By a result of Külshammer [6], B is Morita equivalent to $F^\alpha[D \rtimes I(B)]$ where α is a 2-cocycle in $D \rtimes I(B)$. Hence $\mathbf{Z}B \simeq \mathbf{Z}F^\alpha[D \rtimes I(B)]$ as algebra and

$$\begin{aligned} LL(\mathbf{Z}B) &= LL(\mathbf{Z}F^\alpha[D \rtimes I(B)]) \\ &\leq LL(F^\alpha[D \rtimes I(B)]). \end{aligned}$$

By Lemma 1.2, Proposition 1.5 and Lemma 2.1 in [9], $\mathbf{J}F^\alpha[D \rtimes I(B)] = \mathbf{J}FD \cdot F^\alpha[D \rtimes I(B)] = F^\alpha[D \rtimes I(B)] \cdot \mathbf{J}FD$ and thus $LL(F^\alpha[D \rtimes I(B)]) = LL(FD)$. Moreover, by Theorem (3) in [7], we have $LL(FD) = p^m + p^n - 1$, as claimed. The second part of our claim is clear by the first part. \square

Now we consider the case $p = 2$.

Proposition 2.5. *Assume $D \simeq C_{2^m} \times C_{2^n}$ for some $m, n \geq 1$. Then the one of the following holds:*

- (1) B is Morita equivalent to FD and $LL(\mathbf{Z}B) = 2^m + 2^n - 1$.
- (2) B is Morita equivalent to FA_4 and $LL(\mathbf{Z}B) = 2$.
- (3) B is Morita equivalent to the principal block of FA_5 and $LL(\mathbf{Z}B) = 2$.
- (4) B is Morita equivalent to $F[D \rtimes C_3]$ and $LL(\mathbf{Z}B) \leq 2^{m+1} - 1$. (In this case $2 \leq m = n$)

where A_4 and A_5 are four and five degree alternating groups, respectively.

Proof. Without loss of generality, we may assume $m \geq n$. We first obtain the order of automorphism group $\text{Aut}(D)$ of D as follows.

$$|\text{Aut}(D)| = \begin{cases} 3 \cdot 2^{4m-3} & \text{if } m = n \\ 2^{m+3n-2} & \text{if } m > n. \end{cases}$$

We remark that $e(B)$ divides the odd part of $|\text{Aut}(D)|$ and investigate the following cases.

Case 1 $m > n$.

We have $e(B) = 1$ and thus B is Morita equivalent to FD by [11]. Since $\mathbf{Z}B \simeq \mathbf{Z}FD$ as algebra, we deduce $LL(\mathbf{Z}B) = LL(\mathbf{Z}FD) = LL(FD) = 2^m + 2^n - 1$ by [7].

Case 2 $m = n = 1$.

By the result of Erdmann [2], B is Morita equivalent to FD , or FA_4 or the principal block of FA_5 . In the first case, we have $LL(\mathbf{Z}B) = 3$ as same way to **Case 1**. In the remaining cases, $LL(\mathbf{Z}B) = 2$ by using Proposition 2.2 since $k(B) - l(B) = 1$.

case 3 $m = n \geq 2$.

By Theorem 1.1 in [3], B is Morita equivalent to FD or $F[D \rtimes C_3]$. So we have $LL(\mathbf{Z}B) = 2^{m+1} - 1$ by the same way to **Case 1** or $LL(\mathbf{Z}B) \leq 2^{m+1} - 1$ by Lemma 2.4, respectively. \square

At the end of this section, we study the case that $D \simeq C_3 \times C_{3^n}$ for some $n \geq 1$.

Proposition 2.6. *If $D \simeq C_3 \times C_{3^n}$ for some $n \geq 1$, then $LL(\mathbf{Z}B) \leq 3^n + 2$. In particular, $LL(\mathbf{Z}B) \leq |D| - 4$.*

Proof. We first obtain

$$|\operatorname{Aut}(D)| = \begin{cases} 16 \cdot 3 & \text{if } n = 1 \\ 4 \cdot 3^{n+1} & \text{if } n \geq 2. \end{cases}$$

Case 1 $e(B) \leq 4$.

By results of [11], [12], [13], [14] and Lemma 2.4, we deduce $LL(\mathbf{Z}B) \leq 3^n + 2$.

Since $e(B)$ divides 3'-part of $|\operatorname{Aut}(D)|$, we may assume $n = 1$ in the following.

Case 2 $n = 1$ and $5 \leq e(B)$.

$I(B)$ is isomorphic to one of the following groups:

- C_8 , D_8 (dihedral group of order 8), Q_8 (quaternion group of order 8),
- SD_{16} (semi-dihedral group of order 16).

We first suppose $I(B)$ is isomorphic to D_8 or SD_{16} . By the results of Kiyota [4] and Watanabe [15], $k(B) - l(B)$ is at most 4 and thus $LL(\mathbf{Z}B) \leq 5$ by Proposition 2.2. Finally, suppose $I(B)$ is isomorphic to C_8 or Q_8 . Kiyota [4] has not determined the invariants $k(B), l(B)$ in general. However, we can compute $k(B) - l(B)$ as follows. Since $I(B)$ acts on $D \setminus \{1\}$ regularly, the conjugacy classes for B -subsections are $(1, B)$ and (u, b_u) for some $u \in D \setminus \{1\}$ where b_u is a block of $FC_G(u)$ such that $(b_u)^G = B$. Moreover $I(b_u) \simeq C_{I(B)}(u)$ is trivial by the action of $I(B)$ on $D \setminus \{1\}$, b_u is nilpotent, $k(B) - l(B) = l(b_u) = 1$, and hence $LL(\mathbf{Z}B) = 2$ as claimed.

The last part of the proposition is clear. □

3. PROOF OF THEOREM 1.3

We describe some notations to prove Theorem 1.3. For a p -element x in G , we denote by

$$\begin{aligned} \sigma_x : \mathbf{Z}FG &\rightarrow \mathbf{Z}FC_G(x), \\ \tau_x : FC_G(x) &\rightarrow F[C_G(x)/\langle x \rangle] \end{aligned}$$

the Brauer homomorphism and natural epimorphism, respectively. When $\sigma_x(1_B) \neq 0$ where 1_B is the block idempotent of B , let b_1, \dots, b_r be all blocks of $FC_G(x)$ such that $\sigma_x(1_B)1_{b_i} \neq 0$. For each $1 \leq i \leq r$, $\tau_x(1_{b_i})$ is the unique block idempotent of $F[C_G(x)/\langle x \rangle]$. We put $\bar{b}_i = F[C_G(x)/\langle x \rangle]\tau_x(1_{b_i})$. Now we define two integers as follows.

$$\begin{aligned} \lambda_x &= \max\{LL(\mathbf{Z}\bar{b}_i) \mid 1 \leq i \leq r\}, \\ \lambda &= \max\{\lambda_x(|x| - 1) \mid \sigma_x(1_B) \neq 0\} \end{aligned}$$

where $|x|$ is the order of x . Therefore we prove Theorem 1.3. This proof is inspired by Okuyama [8].

Proof of Theorem 1.3. We may assume that D is not trivial. We follow two steps.

Step 1 We show $LL(\mathbf{Z}B) \leq \lambda + 1$.

Let denote by $Z_{p'}$ the F -subspace of $\mathbf{Z}FG$ spanned by all p -regular section sums. For our claim above, it suffices to prove that $(\mathbf{J}\mathbf{Z}FG)^\lambda 1_B \subseteq Z_{p'}$ since $Z_{p'}$ is contained in the socle of FG and thus $\mathbf{J}\mathbf{Z}FG \cdot Z_{p'} = 0$. Take an element a in the left side and write $a = \sum_{g \in G} a_g g$ where $a_g \in F$. We want to show that $a_g = a_h$ for all $g, h \in G$, if the p -regular parts of them are G -conjugate. However, since $a \in \mathbf{Z}FG$, we need only see that $a_{xy} = a_y$ for all p -elements x and p -regular elements y in G with $xy = yx$. We first suppose $\sigma_x(1_B) = 0$. Then $\sigma_x(a) = 0$ and thus $a_{xy} = a_y = 0$ as required. So we may assume that $\sigma_x(1_B) \neq 0$. Therefore we have

$$\begin{aligned} \sigma_x((\mathbf{J}\mathbf{Z}FG)^\lambda 1_B) &\subseteq \sum_{1 \leq i \leq r} (\mathbf{J}\mathbf{Z}FC_G(x))^\lambda 1_{b_i} \\ &= \sum_{1 \leq i \leq r} (\mathbf{J}\mathbf{Z}b_i)^\lambda \\ &\subseteq \sum_{1 \leq i \leq r} (\mathbf{J}\mathbf{Z}b_i)^{\lambda_x(|x|-1)} \end{aligned}$$

On the other hand, for each $1 \leq i \leq r$, $\tau_x((\mathbf{J}\mathbf{Z}b_i)^{\lambda_x}) \subseteq (\mathbf{J}\mathbf{Z}\bar{b}_i)^{\lambda_x} = 0$ and hence $(\mathbf{J}\mathbf{Z}b_i)^{\lambda_x} \subseteq \text{Ker } \tau_x$. Since $\text{Ker } \tau_x = (x-1)FC_G(x)$, we conclude

$$\begin{aligned} (\mathbf{J}\mathbf{Z}b_i)^{\lambda_x(|x|-1)} &\subseteq \{(x-1)FC_G(x)\}^{|x|-1} \\ &= (x-1)^{|x|-1} FC_G(x) \\ &= (1+x+\dots+x^{|x|-1})FC_G(x). \end{aligned}$$

Thereby we have $\sigma_x(a) \in (1+x+\dots+x^{|x|-1})FC_G(x)$, $x\sigma_x(a) = \sigma_x(a)$ and thus $a_{xy} = a_y$ as claimed.

Step 2 We show $\lambda + 1 \leq p^d - p^{d-\varepsilon} + 1$.

We fix a p -element x in G of order p^{ε_1} and block b of $FC_G(x)$ associated to λ . Namely, $\lambda = LL(\mathbf{Z}\bar{b})(p^{\varepsilon_1} - 1)$. We remark that $0 < \varepsilon_1$ when D is not trivial. Let D_1 be a defect group of b of order p^{d_1} . Then D_1 is contained in D up to G -conjugate, $\varepsilon_1 \leq \varepsilon$ and we may assume the defect group of \bar{b} is $\bar{D}_1 = D_1/\langle x \rangle$ of order $p^{d_1-\varepsilon_1}$. Therefore we have

$$\begin{aligned} LL(\mathbf{Z}\bar{b})(p^{\varepsilon_1} - 1) + 1 &\leq p^{d_1-\varepsilon_1}(p^{\varepsilon_1} - 1) + 1 \\ &\leq p^{d-\varepsilon_1}(p^{\varepsilon_1} - 1) + 1 \\ &= p^d - p^{d-\varepsilon_1} + 1 \\ &\leq p^d - p^{d-\varepsilon} + 1. \end{aligned}$$

The theorem is completely proved. \square

Corollary 3.1. *In the proof of Theorem 1.3, if $LL(\mathbf{Z}\bar{b})(p^{\varepsilon_1} - 1) + 1 = p^d - p^{d-\varepsilon} + 1$, then $D \simeq C_{p^\varepsilon} \times C_{p^{d-\varepsilon}}$. In particular, if $LL(\mathbf{Z}B) = p^d - p^{d-\varepsilon} + 1$, then $D \simeq C_{p^\varepsilon} \times C_{p^{d-\varepsilon}}$.*

Proof. By the inequality in **Step 2** in the proof above, we have $\varepsilon_1 = \varepsilon$ and $d_1 = d$. Moreover $\bar{D}_1 = D_1/\langle x \rangle$ is cyclic by Theorem 1.2. Thus, since $\langle x \rangle$ is contained in the center $Z(D_1)$ of D_1 , $D_1/Z(D_1)$ is also cyclic. This implies D_1 is abelian. Thereby, we have $D \simeq D_1 = \langle x \rangle \times H$ where $H \simeq D_1/\langle x \rangle$. \square

4. LARGE LOEWY LENGTHS

In this last section, we prove Theorem 1.4 and 1.5. We remark that the notations given in the proof of Theorem 1.3 will be used throughout this section.

Proof of Theorem 1.4. In case D is cyclic, we obtain $D \simeq C_3$ and $e(B) = 2$ by Proposition 2.3. So we may assume D is not cyclic and hence $\varepsilon < d$. Then, since $LL(\mathbf{Z}B) = p^d - 1 \leq p^d - p^{d-\varepsilon} + 1 < p^d$, we have $D \simeq C_2 \times C_{2^{d-1}}$ by Corollary 3.1. Furthermore, $d = 2$ and (2) holds by Proposition 2.5 as claimed. \square

Proof of Theorem 1.5. As same reason to the proof of Theorem 1.4, we may assume D is not cyclic, $\varepsilon < d$ and $LL(\mathbf{Z}B) = p^d - 2 \leq LL(\mathbf{Z}\bar{b})(p^{\varepsilon_1} - 1) + 1 \leq p^d - p^{d-\varepsilon} + 1 \leq p^d - 1$.

Case 1 $LL(\mathbf{Z}B) = p^d - 2 = p^d - p^{d-\varepsilon} + 1$.

By Corollary 3.1, we have $D \simeq C_3 \times C_{3^{d-1}}$. However, $LL(\mathbf{Z}B) \neq p^d - 2$ in this case by Proposition 2.6.

Case 2 $LL(\mathbf{Z}\bar{b})(p^{\varepsilon_1} - 1) + 1 = p^d - p^{d-\varepsilon} + 1 = p^d - 1$.

By Corollary 3.1, $D \simeq C_2 \times C_{2^{d-1}}$. Moreover, by Proposition 2.5, (2) or (3) holds.

Case 3 $LL(\mathbf{Z}\bar{b})(p^{\varepsilon_1} - 1) + 1 = p^d - 2$ and $p^d - p^{d-\varepsilon} + 1 = p^d - 1$.

We obtain $p = 2$, $d - \varepsilon = 1$ and $LL(\mathbf{Z}\bar{b}) = \frac{2^d - 3}{2^{\varepsilon_1} - 1}$. Since $LL(\mathbf{Z}\bar{b}) \leq \bar{D}_1 \leq 2^{d-\varepsilon_1}$, $d - \varepsilon_1 = 1$ (remark that $0 < d - \varepsilon \leq d - \varepsilon_1$) and so $LL(\mathbf{Z}\bar{b}) = \frac{2^{\varepsilon_1+1}-3}{2^{\varepsilon_1}-1} = 1$ or 2 . Thus we have $\varepsilon_1 = 1$ and $d = 2$. In this case, (2) or (3) holds by Proposition 2.5. \square

We omit the proof of Theorem 1.6 since we can prove it by similar way to Theorem 1.4 and 1.5.

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YOSHIHIRO OTOKITA:

DEPARTMENT OF MATHEMATICS AND INFORMATICS,
GRADUATE SCHOOL OF SCIENCE,
CHIBA UNIVERSITY,
1-33 YAYOI-CHO, INAGE-KU,
CHIBA-SHI, 263-8522, JAPAN.
E-mail address: otokita@chiba-u.jp